Readers' Forum

Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Methods of Design Sensitivity Analysis in Structural Optimization"

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IN Ref. 1, Arora and Haug present three methods for computing gradients, demonstrating that their own is superior to the other two. Unfortunately, their paper has two drawbacks. First, they chose a nomenclature which is so different from that used by most people in this field that it is almost incomprehensible. Second, by creating something of a science out of chain-rule differentiation, the most significant contribution of their paper may be easily overlooked. That is that the order of matrix operations has a significant effect on computational efficiency.

To clarify this, consider the computation of the gradient of a constraint with respect to a design variable, using more familiar nomenclature.

Finite-element analysis requires the solution of the matrix equation

$$K\bar{u}=\bar{P}$$

where K is the stiffness matrix, $\bar{P}$ is a vector of applied loads, and $\bar{u}$ is a vector of joint displacements. Note that multiple loading conditions are considered simply by adding $\bar{P}$ and $\bar{u}$ vectors. We know K and $\bar{P}$ and must solve for the displacements $\bar{u}$ as

$$\bar{u}=K^{-1}\bar{P}$$

where it is understood that we will not actually invert K, but will use the most efficient linear equation solver at our disposal.

Now in design we define some constraint parameter,

$$g=f(\bar{u},\bar{X}) \tag{1}$$

where $\bar{X}$ is a vector containing the independent design variables. There will usually be many such constraints. One example is a stress constraint

$$g=\sigma_{ij}-\bar{\sigma}$$

where σ_{ij} is the stress in element i under load condition j and $\bar{\sigma}$ is the allowable stress. In general, both σ_{ij} and $\bar{\sigma}$ may be functions of the design variables $\bar{X}$ and the displacements $\bar{u}$ (which are in turn functions of $\bar{X}$).

Taking the partial derivative of g with respect to design variable x_i yields

$$\frac{\partial g}{\partial x_i}=\frac{\partial f}{\partial x_i}+\frac{\partial f^T}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial x_i} \tag{2}$$

where $\partial f/\partial \bar{u}$ and $\partial \bar{u}/\partial x_i$ are vectors of the same dimension as vector $\bar{u}$.

Remembering that $K\bar{u}=\bar{P}$

$$\frac{\partial K}{\partial x_i}\bar{u}+K\frac{\partial \bar{u}}{\partial x_i}=\frac{\partial \bar{P}}{\partial x_i}$$

so

$$\frac{\partial \bar{u}}{\partial x_i}=K^{-1}\left\{\frac{\partial \bar{P}}{\partial x_i}-\frac{\partial K}{\partial x_i}\bar{u}\right\} \tag{3}$$

Substituting this into Eq. (2) gives

$$\frac{\partial g}{\partial x_i}=\frac{\partial f}{\partial x_i}+\frac{\partial f^T}{\partial \bar{u}}K^{-1}\left\{\frac{\partial \bar{P}}{\partial x_i}-\frac{\partial K}{\partial x_i}\bar{u}\right\} \tag{4}$$

If we retain all terms, all three methods presented by Arora and Haug come to the above equation, proving that a gradient is a gradient!

Now consider an operation count on the matrix multiplication here. Let N be the number of displacement degrees-of-freedom, NDV the number of independent design variables, NG the number of constraints g, and NLC the number of separate loading conditions. Now look at the operations for the last term in Eq. (4)

$$\underset{\text{NG}\times\text{N}}{\frac{\partial f^T}{\partial \bar{u}}}\;\underset{\text{N}\times\text{N}}{K^{-1}}\;\left\{\underset{\text{N}\times\text{NLC}}{\frac{\partial \bar{P}}{\partial x_i}}-\underset{\text{N}\times\text{N}}{\frac{\partial K}{\partial x_i}}\;\underset{\text{N}\times\text{NLC}}{\bar{u}}\right\}$$

All components of this matrix operation must be calculated in each of the three methods. Therefore, the *only difference is the sequence in which this matrix operation is performed.* Arora and Haug make a definite contribution in recognizing this fact.

To clarify this further, write the matrix operation in the compact form

$$A^TK^{-1}B$$

The operation may be performed in either of two ways:

$$A^T(K^{-1}B)=A^TC \qquad C \text{ computed from } KC=B$$

or

$$(A^TK^{-1})B=D^TB \qquad D \text{ computed from } KD=A$$

This operation is for the ith design variable, so A and B must be computed NDV times.

Therefore, assuming there is an active constraint in each loading condition, solution for C requires NDV*NLC right-hand sides in the $KC=B$ operation. On the other hand, solution for D requires NG right-hand sides in the $KD=A$ operation.

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If we want gradients for only a few g's, the second approach is clearly most efficient. Arora and Haug make the assumption that NG is always small compared to NDV*NLC. In a real design situation where we are setting up an approximate design problem, either for hand calculations or for optimization, this may not be true since NG will be the number of constraints *retained for design consideration*. In any case, the message is that the order of the matrix operation should be determined at the time it is done, based upon a comparison of NG and NDV*NLC and the computational cost of each ordering. In terms of *total* computational effort to analyze the structure and compute gradients, the time difference should not be as significant as we are led to believe.

Based upon the discussion, the following conclusions seem to be in order;

1) A gradient is a gradient, no matter how it is calculated. In suggesting that their method is more general, Arora and Haug are only saying that they retain more terms. Certainly the referenced investigators are qualified to do the same if they need the additional terms.

2) The conclusion that their method is up to ten times faster is valid so long as we understand that they are talking about one *part* of the total computational effort.

3) It is important to remember that the order of a matrix operation is criticial to the computational efficiency of performing that operation.

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Reply by Authors to G.N. Vanderplaats

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In responding to Vanderplaats' Comment, we would first like to agree with his operations count and the observation that the order of matrix computation can lead to a computational advantage of up to a factor of ten for the state space method over the design space method for "one part of the computational effort." It is worth noting that for many constraints, such as displacement and stress constraints for constant strain elements, this "one part of the computational effort" is the only computational effort needed for gradient calculations.

There are some points in Vanderplaats' Comments with which we disagree. First, the main point of our paper was that the state space method, which has been a principal tool of optimal control theory for two decades,[1,2] unifies design sensitivity analysis in the field of structural optimization. Our paper shows that the excellent design derivative analysis method of Venkayya and co-workers[3-9] can be viewed as a special case of the adjoint variable, state-space method of design sensitivity analysis. Thus, the unified state space method developed by Bryson[2] for optimal control and adopted by the writers[10] provides a general approach for design sensitivity analysis in a wide variety of structural and mechanical system design problems. Vanderplaats' interpretation of the state space method of design sensitivity analysis as "a science of chain rule differentiation" is unfortunate and worthy of clarification.

In addition to being directly applicable for static response problems, which was the focus of our paper, the method leads directly to an algorithm for design sensitivity analysis and optimization of structures and vibration isolators under dynamic loads.[11-13] The method is also ideally suited for incorporation with the substructuring method of structural analysis.[10,14] More recently, the method has been applied for design sensitivity analysis of large-scale, nonlinear mechanisms and machines.[15] Finally, the state space method is directly applicable to structures whose displacement is the solution of partial differential equations.[10,13,16] For such distributed parameter problems, the state space approach outlined in our paper is used with integral scalar product for design sensitivity analysis of systems described by differential equations, which generalize the matrix method of our paper. Thus, the state space method we present carries over to general structural and mechanical systems described by differential equations, whereas the direct differentiations used in the design space method are meaningless or impractical. Thus, much as in the case of optimal control theory,[2] there is indeed a systematic theory of design sensitivity analysis of broad classes of structural and mechanical systems, which is hardly as simplistic as a "science of chain-rule differentiation."

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Index categories: Structural Design; Analytical and Numerical Methods; Structural Statics.

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